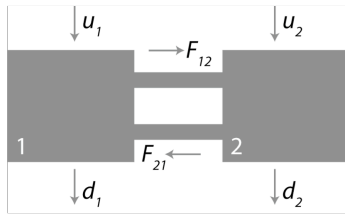


Team problem: compartment model (two compartment case)



Very common in biology/ecology, e.g. an organ and blood in circulation, transmembrane exchange, protein transport, population dynamics and migration, infectious diseases...

Problem description:

- Two substances (e.g. drug, hormone, or proteins etc.), with concentrations x_1 and x_2 (mass/vol) and masses m_1 and m_2 , respectively.
- The compartments have volumes V_1 and V_2 .
- Flow (vol/sec) from compartment 1 to compartment 2 (F_{12}), and reverse flow (F_{21}).
- When a substance happens to be in a compartment, a fraction $d_i \Delta t$ of its mass degrades (or is consumed) in any small time interval Δt .
- There is also an external source (externally injected) and we let u_i denote the inflow (mass/sec) into the compartment i .

Task:

- Write differential equations for the rate of change of masses m_1 and m_2 over time.

[20 MINUTES]

Solution:

- On a small time interval Δt the change of the mass of the substance (m_1) in the first compartment is:

$$m_1(t + \Delta t) - m_1(t) = -F_{12}x_1\Delta t + F_{21}x_2\Delta t - d_1m_1\Delta t + u_1\Delta t.$$

For example, the mass flowing from compartment 1 to compartment 2 is computed as:

$$\text{flow } (F_{12}) \times \begin{array}{l} \text{concentration of} \\ \text{the substance} \end{array} (x_1) \times \text{time } (\Delta t) = F_{12}x_1\Delta t \frac{\text{vol}}{\text{time}} \times \frac{\text{mass}}{\text{vol}} \times \text{time} = F_{12}x_1\Delta t \text{ mass.}$$

- Using the fact that concentration=mass/volume, i.e. $x_1=m_1/V_1$ and $x_2=m_2/V_2$ we get

$$m_1(t + \Delta t) - m_1(t) = -F_{12} \frac{m_1}{V_1} \Delta t + F_{21} \frac{m_2}{V_2} \Delta t - d_1m_1\Delta t + u_1\Delta t.$$

- Dividing by Δt

$$\frac{m_1(t + \Delta t) - m_1(t)}{\Delta t} = -F_{12} \frac{m_1}{V_1} + F_{21} \frac{m_2}{V_2} - d_1m_1 + u_1.$$

- Taking the limit $\Delta t \rightarrow 0$ gives a differential equation describing the change in m_1 over time:

$$\frac{dm_1}{dt} = -F_{12} \frac{m_1}{V_1} + F_{21} \frac{m_2}{V_2} - d_1 m_1 + u_1.$$

- Similarly, one can derive an equation describing the change in m_2 over time, namely:

$$\frac{dm_2}{dt} = F_{12} \frac{m_1}{V_1} - F_{21} \frac{m_2}{V_2} - d_2 m_2 + u_2.$$

- Reducing the number of parameters: following the procedure outlined in the lecture notes, we write:

$$m_1 = \hat{m}_1 m_1^*, \quad m_2 = \hat{m}_2 m_2^*, \quad t = \hat{t} t^*,$$

and substitute:

$$\begin{aligned} \frac{d(\hat{m}_1 m_1^*)}{d(\hat{t} t^*)} &= -F_{12} \frac{\hat{m}_1 m_1^*}{V_1} + F_{21} \frac{\hat{m}_2 m_2^*}{V_2} - d_1 \hat{m}_1 m_1^* + u_1, \\ \frac{d(\hat{m}_2 m_2^*)}{d(\hat{t} t^*)} &= F_{12} \frac{\hat{m}_1 m_1^*}{V_1} - F_{21} \frac{\hat{m}_2 m_2^*}{V_2} - d_2 \hat{m}_2 m_2^* + u_2. \end{aligned}$$

Using $\frac{dm_1}{dt} = \frac{d(\hat{m}_1 m_1^*)}{d(\hat{t} t^*)} = \frac{m_1^*}{t^*} \frac{d\hat{m}_1}{d\hat{t}}$ & $\frac{dm_2}{dt} = \frac{d(\hat{m}_2 m_2^*)}{d(\hat{t} t^*)} = \frac{m_2^*}{t^*} \frac{d\hat{m}_2}{d\hat{t}}$, simplifying and collecting terms we get

$$\begin{aligned} \frac{d\hat{m}_1}{d\hat{t}} &= -\frac{F_{12} t^*}{V_1} \hat{m}_1 + \frac{F_{21} m_2^* t^*}{V_2 m_1^*} \hat{m}_2 - d_1 t^* \hat{m}_1 + \frac{u_1 t^*}{m_1^*}, \\ \frac{d\hat{m}_2}{d\hat{t}} &= \frac{F_{12} t^*}{V_1} \hat{m}_1 - \frac{F_{21} m_2^* t^*}{V_2 m_1^*} \hat{m}_2 - d_2 t^* \hat{m}_2 + \frac{u_2 t^*}{m_2^*}. \end{aligned}$$

It would be nice, for example, to make:

$$d_1 t^* = 1 \quad \text{and} \quad \frac{u_1 t^*}{m_1^*} = 1$$

and it seems a reasonable assumption that $m_1^* = m_2^*$. This can indeed be done, by setting:

$$t^* = \frac{1}{d_1} \quad \text{and} \quad m_1^* = m_2^* = \frac{u_1}{d_1}.$$

To give

$$\begin{aligned} \frac{d\hat{m}_1}{d\hat{t}} &= -\frac{F_{12}}{d_1 V_1} \hat{m}_1 + \frac{F_{21}}{d_1 V_2} \hat{m}_2 - \hat{m}_1 + 1, \\ \frac{d\hat{m}_2}{d\hat{t}} &= \frac{F_{12}}{d_1 V_1} \hat{m}_1 - \frac{F_{21}}{d_1 V_2} \hat{m}_2 - \frac{d_2}{d_1} \hat{m}_2 + \frac{u_2}{u_1}. \end{aligned}$$

Or, dropping the hats and introducing the new constants:

$$\alpha_1 = \frac{F_{12}}{d_1 V_1}, \quad \alpha_2 = \frac{F_{21}}{d_1 V_2}, \quad \alpha_3 = \frac{d_2}{d_1} \quad \text{and} \quad \alpha_4 = \frac{u_2}{u_1},$$

we end up with:

$$\begin{aligned}\frac{dm_1}{dt} &= -\alpha_1 m_1 + \alpha_2 m_2 - m_1 + 1, \\ \frac{dm_2}{dt} &= \alpha_1 m_1 - \alpha_2 m_2 - \alpha_3 m_2 + \alpha_4.\end{aligned}$$

[ADDITIONAL EXERCISE]: Analyze the linear stability of this system.

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